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ORIGINAL RESEARCH PAPER

Assessing Prospective Teachers' Geometric Transformations Thinking: A Van Hiele Theory-Based Analysis with Eye Tracking Cognitive Science Method

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ABSTRACT

Background and Objectives: Geometric transformations have played a crucial role throughout history in various aspects of human life. Symmetry is one of the important concepts in school mathematics. Students' academic performance is intricately connected to the knowledge and skills of their educators. Recognizing the importance of prospective teachers (PTs) as future educators, in the initial stage, the aim of this research is to assess and analyze the levels of geometric thinking among prospective elementary teachers (PETs) utilizing Van Hiele's theory. Subsequently, the research seeks to delve into the thinking process and gaze patterns of prospective mathematics education teachers (PMETs) using the cognitive science method of eye tracking.

Materials and Methods: This study focuses on investigating and evaluating the thinking of geometric transformations and problem-solving skills among prospective teachers (PTs). The research method employed a combined survey method, encompassing two distinct tests conducted on two groups of PTs. The accessible statistical sample includes 50 participating PETs and 21 participating PEMTs from Iran. The PETs of Farhangian University of Isfahan were divided into two groups: 42 students who had not learned the concept of geometric transformations in their undergraduate program (NPGT), and 8 students who had learned this concept in their undergraduate program (PGT). To assess the level of geometric thinking among participants, a self-made geometric test based on Van Hiele's theory was utilized. The test reliability was assessed using Cronbach's alpha coefficient, which yielded a value of 0.68. Additionally, the validity of the test has been confirmed by some professors. In evaluating geometric thinking, a cognitive science method was performed. This method involved designing a psychophysical experiment and recording eye movements of the PMETs. The psychophysical experiment part was conducted in the computer laboratory of Shahid Rajaee Teacher Training University, Tehran, and was performed by Eyelink device and MATLAB software on student teachers of mathematics education of this university.

Findings: The results of the research show that students recognize the shape with symmetry as a symmetrical shape, but they perform poorly in determining the type of symmetry of symmetrical shapes, especially when a shape has rotational symmetry or oblique axial symmetry or a combination of several types of symmetry. In the first stage, the evaluation of PETs responses showed that 34% of them were in the first level and 18% in the second level of Van Hiele. The cognitive findings revealed that PMETs demonstrated superior performance in recognizing symmetries characterized by a single type of symmetry, in contrast to shapes involving combinations of various symmetries. Examining the recorded eye-tracking images of the students revealed a difference in gaze patterns between the groups that gave correct and incorrect answers. In addition, this difference is also evident among images with different symmetries (reflection, central, rotational).

Conclusions: The current research confirms the weakness of students in identifying the type of symmetry in symmetrical shapes. It also emphasizes the need to pay more attention to the training of PTs during their academic years. To address this, it is suggested to revise the curriculum concerning geometric transformations in the university courses for PTs training, additionally, the utilization of software such as Augmented Reality (AR) and GeoGebra can contribute to enhancing cognitive and visual abilities of PTs in comprehending the concept of symmetry.

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64

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8

مقاله پژوهشی

ارزیابی تفکر تبدیلات هندسی دانشجو معلمان: تحلیل مبتنی بر نظریه ون هیلی با استفاده از روش علوم شناختیِ ردیاب چشم

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چکیده

پیشینه و اهداف: تبدیلات هندسی در طول تاریخ نقش مهمی در جنبه‌های مختلف زندگی بشر داشته‌اند. تقارن یکی از مفاهیم مهم در ریاضیات مدرسه است. واضح است که عملکرد تحصیلی دانش‌آموزان به طور پیچیده‌ای با دانش و مهارت‌های مربیان آن‌ها مرتبط است. با توجه به اهمیت نقش دانشجو معلمان به عنوان مربیان آینده، در مرحله اول، هدف این تحقیق بررسی و ارزیابی سطوح تفکر تبدیلات هندسی در بین دانشجو معلمان ابتدایی بر اساس نظریه یادگیری ون هیلی است. پس از آن، این تحقیق به دنبال بررسی فرآیند تفکر و ارزیابی الگوهای نگاه دانشجو معلمان آموزش ریاضی به عنوان گروه متخصص، با استفاده از روش علوم شناختیِ ردیاب چشم است.

روش‌ها: این مطالعه به بررسی و ارزیابی تفکر تبدیلات هندسی و مهارت‌های حل مسئله در میان دانشجو معلمان متمرکز است. روش تحقیق مورد استفاده پیمایشی آمیخته است. نمونه آماری در دسترس شامل ۵۰ دانشجو معلم ابتدایی و ۲۱ دانشجو معلم آموزش ریاضی ایرانی است. دانشجو معلمان ابتدایی از دانشگاه فرهنگیان اصفهان انتخاب شده و به دو گروه شامل ۴۲ نفر از دانشجویانی که مفهوم تبدیلات هندسی را در مقطع کارشناسی نیاموخته بودند و ۸ نفر از دانشجویانی که این مفهوم را در مقطع کارشناسی آموخته بودند، تقسیم شدند. برای بررسی سطح تفکر هندسی شرکت‌کنندگان، از آزمونی محقق ساخته بر اساس نظریه ون هیلی استفاده شد. پایایی آزمون با استفاده از ضریب آلفای کرونباخ ارزیابی شد که مقدار ۰.۶۸ بدست آمد. روایی آزمون نیز توسط اساتید مورد تایید قرار گرفت. در ارزیابی تفکر هندسی، یکی از روش‌های علوم شناختی به کار گرفته شد. این روش شامل طراحی آزمون روان-فیزیک همراه ثبت حرکات چشم در گروه دانشجو معلمان آموزش ریاضی بود. آزمون روان-فیزیک در آزمایشگاه کامپیوتر دانشگاه تربیت معلم شهید رجایی تهران با دستگاه Eyelink و نرم افزار MATLAB بر روی دانشجو معلمان آموزش ریاضی این دانشگاه، انجام شد.

یافته‌ها: یافته‌های تحقیق نشان داد که دانشجویان، شکل با تقارن را به عنوان شکل متقارن تشخیص می‌دهند، اما در تعیین نوع تقارن اشکال متقارن عملکرد ضعیفی دارند، به ویژه زمانی که یک شکل دارای تقارن چرخشی یا تقارن محوری مایل یا ترکیبی از چندین نوع تقارن باشد. ۳۴٪ از دانشجو معلمان آموزش ابتدایی در سطح اول و ۱۸٪ در سطح دوم نظریه ون هیلی قرار گرفتند. یافته‌های شناختی نشان داد که دانشجو معلمان آموزش ریاضی در تشخیص اشکال متقارنی که با یک نوع تقارن مشخص شده‌اند نسبت به شکل‌هایی که شامل ترکیبی از تقارن‌های مختلف هستند، عملکرد نسبتا موفق‌تری داشته‌اند. بررسی تصاویر ثبت‌شده ردیابی چشم دانشجویان، تمایزی را در الگوهای نگاه، بین گروه‌هایی که پاسخ‌های درست و نادرست در ارائه می‌دادند، نشان داد. علاوه بر این، این تفاوت در بین تصاویر با تقارن‌های مختلف(خطی، مرکزی، چرخشی) نیز مشهود است.

نتیجه‌گیری: پژوهش حاضر ضعف دانشجو معلمان در شناسایی نوع تقارن در اشکال را تائید می‌کند. همچنین بر نیاز به توجه بیشتر به امور برای رسیدگی به این موضوع، پیشنهاد می‌شود که برنامه درسی مربوط به تبدیلات هندسی در دوره آموزشی دانشگاه دانشجو معلمان مورد بازنگری قرار گیرد. علاوه بر این، استفاده از نرم‌افزارهایی مانند واقعیت افزوده و جئوجبرا می‌تواند به افزایش توانایی‌های شناختی و بصری دانشجو معلمان در درک و تشخیص مفهوم تقارن کمک کند.

Introduction

Mathematics application in real-life situations is an essential aspect of the subject, and geometric transformations represent one of its most significant applications. Symmetry, as a key concept in geometry and the foundational principle of mathematical concepts [1], is a fundamental part of geometry and nature, creating patterns that help us conceptually organize our world [2]. The concept of symmetry is the one that humans can intuitively recognize [3]. It is often linked with aesthetic beauty and can be observed in both natural phenomena and human-made objects, such as artworks and manufactured products, across the globe [4]. It is widely acknowledged that symmetry is a crucial element in beauty in art and design. For instance, the Birkhoff aesthetic measure, developed in two probabilistic and sequential dimensions, accurately evaluates symmetry in designs and artistic effects. This measure is based on the principles of harmony, balance, simplicity, and complexity, and aims to capture subjective beauty experiences in objective terms [5]. Symmetry also enables students to visualize various geometric concepts and relate them to real-life experiences [6]. Moreover, it provides many opportunities for interdisciplinary reinforcement [7]. Therefore, symmetry is one of the most significant mathematics applications. The concept of geometric transformation is easy to grasp, and it can be illustrated through many real-world examples [8], making it crucial to teach it in elementary schools [2]. Mathematics teachers emphasize the need for students to acquire this concept at a young age [9]. Research in this field has highlighted common misconceptions among students, including their difficulty in distinguishing between axial and central symmetry [10, 11].

The background of this research suggests known issues with teaching the concepts of symmetry, rotation, translation, and geometric transformations in general in elementary and high schools [12]. In a study conducted by Zaslavsky [13], it was revealed that students' challenges in understanding symmetry are closely linked to teachers' misconceptions about symmetry. According to research, the symmetry and rotation skills of prospective elementary mathematics teachers have a significant impact on their ability to teach mathematical concepts effectively [14]. Specifically, teachers with a deep understanding of symmetry and rotation are more likely to:

- Help their students gain strong spatial reasoning skills: Spatial reasoning is essential for understanding mathematical concepts. Teachers who can help their students develop strong symmetry and rotation skills can also help them develop their spatial reasoning abilities.
- Help their students make connections between different mathematical concepts: Symmetry and rotation are closely related to other mathematical concepts, such as geometry and algebra. Teachers who understand these concepts can help their students make connections between different mathematics topics.
- Use visual aids effectively: Symmetry and rotation are visual concepts that can be easily demonstrated through diagrams and other visual aids. Teachers with strong symmetry and rotation skills can use these tools effectively to help their students understand complex mathematical ideas.

In summary, symmetry and rotation skills help students develop strong spatial reasoning skills. This includes making connections between different mathematical concepts and using visual aids effectively.

As reported by Reyhani et al. [15] undergraduate geometry courses may not adequately equip students with the knowledge necessary for effective geometry teaching in high school. Therefore, considering the importance and necessity of the concept of symmetry, and the belief that teachers play an essential role in presenting quality mathematics to students, teachers need to know the mathematical structure of symmetry.

Providing training for elementary school teachers, especially during their teacher education program, is of critical importance. When something is taught to a teacher, a certain level of proficiency can be expected. Therefore, any shortcomings in elementary school teachers' training, especially during their teacher training program, will inevitably result in irreparable damages.

Given that most of research related to geometry emphasizes the weakness of students in geometry and has not paid attention to the root of this weakness, which mostly dates back to elementary school [16], and previous literature review has not shown similar studies in Iran, this study aims to evaluate the geometric thinking of PETs at the first two levels of the van Hiele, focusing on the topic of geometric transformations. The Van Hiele theory describes geometric thinking, consisting of five stages: Visualization, Analysis, Abstraction, Deduction, and Rigor [17, 18]. As the fourth and fifth levels of this theory extend beyond PETs knowledge and the topics covered in elementary school textbooks, they are not evaluated in this research.

Today, cognitive science is important as an interdisciplinary knowledge, and with the advancement of technology, appropriate tools and methods have been provided in this field that can be used without the direct involvement of the sample in behavioral and cognitive research [19]. Part of the current research relies on the eye movement detection tool, which is one of the tools used in cognitive science [20].

The aim of this study is to identify the geometric thinking of PETs in performing tasks related to geometric transformations, including reflection symmetry, central symmetry, and rotational symmetry. In this regard, a framework based on Van Hiele levels is used to evaluate problem-solving processes, and the main goal is to determine the skill level at which PETs solve geometric transformation problems. In addition, this study focuses on examining participants' eye movements and uses eye tracking tools to examine their performance in identifying types of symmetry and selecting symmetrical shapes. The current research seeks to answer the following research questions:

- What is the performance of PETs in providing answers evaluated at the first and second levels of Van Hiele?
- Have curriculum changes in elementary education at the undergraduate level improved PETs' geometric transformation thinking levels?
- How is the cognitive performance of PMETs in selecting symmetrical images?
- Is there a significant difference between visually recorded images of correct and incorrect PMETs responses?

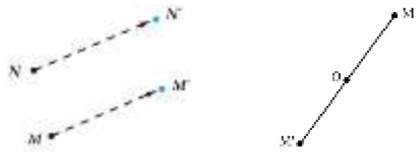
Review of the Related Literature

Theoretical foundations and research literature in the fields of mathematics education and cognitive sciences are explained as follows.

Theoretical foundations

Geometric transformations

Geometric transformations were first introduced in a seminar entitled “The Erlangen Program by Christian Felix Klein” in 1872. Klein defined geometry as objects whose properties remain constant under transformations [25]. In Dodge's study [26], a transformation in a plane is defined as a one-to-one correspondence from a plane to itself. Suppose that each point N on the plane is displaced to a new position N´ within the same plane. In this scenario, N´ is deemed as the image of N, whereas N is acknowledged as the preimage of N´. In a plane, such as P, a transformation T is a function such that every point M of P corresponds to exactly a point M´ in the same plane, with the property that T(M)=M´ (Figure 1. (a)). A one-to-one correspondence of the plane onto itself is established if only distinct points have distinct images, and each point within the plane has a unique preimage point. This process is referred to as a transformation of the plane [27, 28].



(a) (b)

Fig. 1: (a) Transformation of the Plane. (b) The point M´ is the symmetric point of M.

A transformation that moves an object from one space to another without changing its size or shape is called a translation [29].

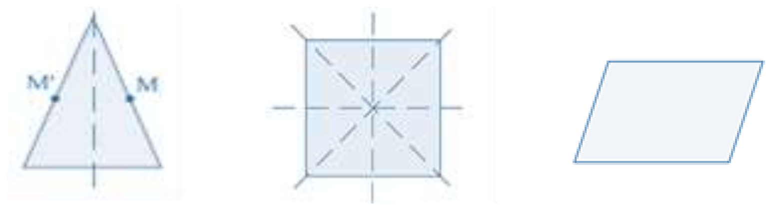
Symmetry

Symmetry of an object refers to the rigid motion (that is, a motion that preserves distance and size) of a plane that does not alter the object [30]. Geometric transformations define symmetry as an isometry. Aksoy et al. [9] describe symmetry as the positioning of a geometric figure or mathematical object in the same or different plane or space while maintaining its nature and characteristics under reflection, rotation, and translational movements. They outline the fundamental components of the symmetry concept, which require the following conditions: 1) the existence of a geometric or mathematical object, 2) performing reflection, rotation, and translational movements on the object, and 3) positioning a version of the original object in a new plane or space that remains unchanged.

According to Long et al. [27], a plane shape exhibits symmetry if any rigid motion of the plane relocates all the points within the shape back to their original points. On the other hand, a transformation of the plane is deemed as a rigid motion when and only when the distance between any two points N and M equals the distance between their respective images' points N´ and M´. In other words, NM = N´M´ holds for all points N and M, as shown in Figure 1. Rigid motions are alternatively referred to as isometries, which means "same measure" in that iso denotes "same" and metric denotes "measure".

Reflection symmetry

As described by Long [27], reflections or flips are classified as one of the four fundamental rigid motions of the plane. Specifically, in a reflection, all points M are moved to their mirror images M´, which are located on the opposite side of a given line while maintaining the same distance from that line (Figure 2).



(a) (b) (c)

Fig. 2: Plane figures and their lines of symmetry. (a)Isosceles triangle:1 line, (b)Square: 4 lines, (c) Parallelogram: no lines of symmetry.

Rotational symmetry

A rotation or a turn is classified as one of the four essential rigid motions of the plane. In a rotation, one specified point acts as the center of rotation, and the other points within the plane are turned or spun around the center at the same angle and direction. This action constitutes a transformation of the plane that upholds both the distances between the points as well as the angles between lines or segments [27].

A figure exhibits rotational symmetry if it can be rotated by an angle between 0° and 360° such that it remains unchanged. Fig. 3 shows an investigation of rotation symmetry for an equilateral triangle. The triangle undergoes a counterclockwise rotation third of a complete revolution in Fig. 3. It is also possible to achieve a matching image through a rotation two-thirds of a complete revolution or a full 360° rotation. Any figure can be rotated completely around any point as the center of rotation to yield a matching image. Figures that only produce an identical image with a full 360° rotation do not exhibit rotational symmetry [31].

Central symmetry

Central symmetry in a two-dimensional plane, also referred to as point symmetry or point reflection, constitutes a distinctive occurrence of rotation within the plane, spanning 180° around a central point. Additionally, it can be formed through the combination of two axial symmetries characterized by mutually perpendicular axes [32]. The central symmetry can be defined with a specific point O in a plane. A transformation that maps any point M to its mirror image with respect to point O is referred to as central symmetry or the center of the turn (see Figure 1. (b)). Therefore, point O is the center of this symmetry, as discussed by [33]. It is imperative to note that central symmetry in a plane is a special case of rotation, which involves rotating the figure 180 degrees around a center point, resulting in its image being an exact reflection of the original [32].

Van Hiele's theory

Pierre-Marie van Hiele (1909-2010) and Dina van Hiele Godolf (1911-1985), who served as math teachers in Montessori schools in the Netherlands, are known for developing a framework aimed at enhancing geometric reasoning. Van Hiele's theory proposes that by facilitating specific educational experiences, the learner can progress through five discrete levels that rely on successful acquisition at each stage [34]. As Reyhani has stated, this theory outlines the different stages of geometric thinking students pass through. It begins with basic recognition and ends with a precise and structured geometric proof [15]. Meanwhile, the literature [35] describes Van Hiele surfaces' distinctive characteristics in geometric transformation contexts, as given in Table 1.

Cognitive Science

The expression Cognitive Science was first used by Christopher Longuet Higgins (1973), a scholar who moved from Chemistry and Theoretical Physics to Artificial Intelligence (AI) [36]. In 1967 he founded a Machine Intelligence and Perception Department in Edinburgh, where he pursued artificial vision. He also created a group of psychologists, linguists, and neuroscientists who worked on interdisciplinary projects. He considered AI a sort of 'theoretical psychology' [36] and became a professor of Experimental Psychology. He aimed to uncover fundamental principles of human cognition, understanding, and perception. His pioneering ideas laid the groundwork for a more profound exploration of the human mind, unlocking new avenues for research within Cognitive Science [37].

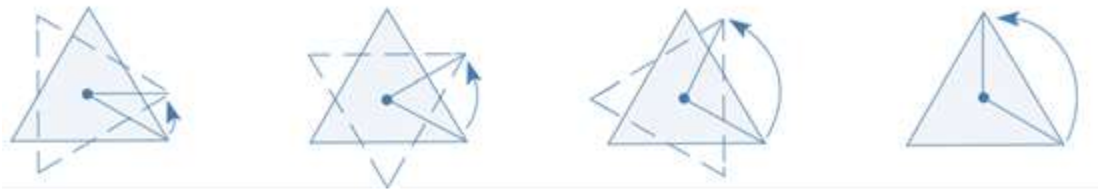


Fig. 3: Rotation symmetry for an equilateral triangle.

Table 1: Van Hiele surfaces' distinctive characteristics in geometric transformation.

Levels	Characteristics: The student …
Level 1	○ identifies transformations by changes in shape; (a) in simple drawings of shapes and images; and (b) in real-life applications images. ○ discerns alteration through actual movement; names, and distinguishes transformations. ○ uses both standard and non-standard labels and names to describe transformations. ○ relies on changing shapes or movements to solve problems instead of using properties of transformations.
Level 2	○ uses attributes of transformations to draw an image or pre-image of a specific transformation. ○ identifies the properties of shape changes resulting from a specific transformation. ○ uses appropriate terminology to describe the properties and transformations involved in geometric thinking. ○ capable of finding the axis of reflection, translation vector, center of rotation, and center of enlargement. ○ correlates transformations using coordinates. ○ applies the well-known properties of geometric transformations to solve problems.
Level 3	○ carries out a combination of simple transformations. ○ explains changes to states (pre-image, image) resulting from composite transformations. ○ student illustrates transformations through the use of coordinates and matrices. ○ correlates the characteristics of transformations with modifications to a shape. ○ can name a transformation based on the initial and final states. ○ given the initial and final states, capable of decomposing and recombining a transformation into a series of simpler transformations.
Level 4	○ provides geometric proofs using a transformation-based method. ○ proves using the coordinates and matrices. ○ demonstrates their ability to solve multi-step problems and provide justifications for their solutions.
Level 5	○ knows the role that the associative, commutative, inverse, and identity properties play in composite transformation operations. ○ recognizes groups of transformations. ○ proves or disproves the subgroups of transformations from group structures.

Cognitive science is an interdisciplinary field of study that focuses on the systematic investigation of cognitive and mental processes in humans and other organisms [38]. It examines and analyzes issues related to perception, thinking, memory, language, visualization, and other cognitive functions. Cognitive science employs various methods and tools, such as experimental psychology, neuroscience, artificial intelligence, linguistics, and the philosophy of mind, to better understand and explain human cognitive processes [39, 40].

Eye Movement

Cognition, as a branch of cognitive sciences, examines the mental processes of humans and their decision-making [41]. The use of cognitive science methods, including eye-tracking tools, enables us to closely investigate human performance in recognizing and selecting symmetrical shapes [19].

Human vision is an active and dynamic process in which the viewer actively seeks specific visual input to support ongoing cognitive and behavioral activities [42, 43]. Eye movements play a crucial role in representing the processing activities of the human mind, including image scanning and cognitive visual-motor activity analysis [44].

For the analysis of the gathered data, it is vital to first establish the definitions of the eye movement indicators presented in the following Table.

Table 2: Summary of key terms

Terms	Description
Fixation	is a brief period during which the eye stops at a fixed point in the visual field [45].
Saccades	Rapid eye movements occur as the gaze moves between different fixation points.
Scan paths	Gaze positions and eye movements are plotted on the stimulus image [46].
Heat map	Gaze positions are plotted on the fixation areas [46].

Related studies

Philosophers and mathematicians have studied symmetry since ancient times [47]. Symmetry has been a significant concept throughout history, and it gained a central role in shaping the scientific vision of the world during the scientific revolution of the 16th century [48]. Symmetry plays a significant role in mathematics and the natural sciences, particularly in physics. For example, during the time of Lorentz and Einstein, symmetries were mostly considered mathematical curiosities that were highly valuable to crystallographers, but not considered fundamental laws of nature [49] or one fundamental concept in physics is the principle of symmetry, which postulates that nature's laws maintain their invariance under certain transformations [50].

Previous research conducted worldwide has indicated that PETs often have limited skills when it comes to symmetry and rotation, which helps them produce inaccurate drawings that do not fulfill the desired level of knowledge. Ignoring these challenges during the teacher training period will result in incomplete education during the teaching course.

Law [51] conducted a study on the ability of PETs to perform geometric transformations, exploring how they learn and process their knowledge. During a geometry course, 18 prospective teachers were asked to define and provide examples of geometric transformations. The researcher observed that these individuals struggled to define transformations accurately, using objective or abstract terms. Similarly, in a recent study [52], prospective teachers struggled to identify the center of rotation when asked to describe a figure's rotation around a point. It has also been reported that elementary students encounter difficulty in identifying rotation centers [14, 53]. Sometimes, these prospective teachers noted that they rotated figures at specific angles, even though these angles were limited to 90, 180, 270, and 360 degrees. Other studies confirm that while reflection symmetry is more straightforward, rotational symmetry is challenging [54-56]. Furthermore, according to the literature [57], students encounter challenges in understanding rotation angles.

It was observed that only 44.4% of teachers could accurately rotate an object and find its center of rotation [58]. This was due to the difficulties they faced with the concept of the center of rotation. Hence, to accurately define the concepts of translation, reflection, and rotation during teacher training and address the associated challenges, researchers [52] suggest that it is very important to involve more students in undergraduate courses related to mathematics education. About the studies conducted on van Hiele's levels of geometric thinking, the main results of the study on pre-service teachers at E. P. College of Education in Bimbilla, Ghana, were that a large proportion of the pre-service teachers demonstrated lower-level geometric thinking abilities. Specifically, many of the participants showed some levels of geometric thinking at the first and second van Hiele levels, which involve recognition and visualization of basic geometric shapes and their properties. Relatively few participants reached levels three, four, and five, which involve using deductive reasoning to make connections between geometric concepts and develop a more sophisticated understanding. The study identified a need for improvements in pre-service math teacher education programs in Ghana to better prepare teachers for teaching geometry and promoting higher-level geometric thinking among students [59]. The summary of research articles in this field indicates that they explore the van Hiele model of geometric thinking and its potential to improve mathematics education. The results of these studies are significant and can be summarized as follows:

Firstly, the papers provide a clear and concise explanation of the five levels of geometric thinking in the van Hiele model. This understanding can help educators assess and teach their students better. Secondly, the importance of recognizing a students' current level of geometric thinking before introducing new concepts is emphasized in these papers. By doing so, educators can tailor their teaching methods to suit the students' needs, resulting in more effective learning. Thirdly, the papers also provide real-world examples of how geometric thinking can be applied in fields such as architecture and engineering, making learning more practical and motivating for students. Finally, the potential for interdisciplinary learning is highlighted in these papers, indicating how the van Hiele model can be integrated into various subjects leading to a well-rounded education for students. Furthermore, these studies paper highlight the significance of the van Hiele model of geometric thinking in mathematics education and offer valuable insights into how educators can use this model to enhance student learning.

On the other hand, the study conducted by Reyhani et al. [15] indicates that the participants demonstrated competence in van Hiele's third level of informal inference. However, they fell short of reaching the fourth level of formal inference. It is evident from these results that the participant's proficiency falls below the expectations set for mathematics teachers and students at the fourth level. This suggests that the current geometry lessons in the undergraduate program of mathematics education may not adequately equip students with the knowledge required to teach high school geometry effectively.

Nowadays, numerous studies have been conducted using cognitive science methods. For example, research entitled 'The Role of Symmetry in the Aesthetics of Residential Building Façades Using Cognitive Science Methods' [19] utilized eye movement recording tools to demonstrate that symmetric patterns attract more attention than asymmetric patterns [approximately 72% of participants chose structures with symmetry lines as aesthetically pleasing]. Additionally, in the displayed images of symmetrical facades (with horizontal and vertical axes), the participants' attention was more focused on the axis of symmetry than other points. Similarly, in this research, it was aimed to evaluate PETs responses in selecting various symmetric images at the Analysis level of the Van Hiele model using cognitive science methodology.

Method

Participants

The participants in this research consisted of 50 female PETs at Fatemeh Al-Zahra campus of Farhangian University in Isfahan and 21 male and female PMETs at Shahid Rajaee Teacher Training University in Tehran.

The PETs of Farhangian University of Isfahan were divided into two groups: 42 students who had not learned the concept of geometric transformations in their undergraduate program (NPGT), and 8 students who had learned this concept in their undergraduate program (PGT).

Instruments

The measurement tool used in this study consisted of two tests, which were developed by the researchers. The details of each test are explained below:

Test questions of PETs

The first question involved identifying three symmetrical figures among five figures, representing Van Hiele's first level. The second question involved evaluating the correctness of four statements related to the concepts and characteristics of rotation, center, and axis of symmetry, which represent the second level of Van Hiele (Fig. 4). The correct answers for the test were determined based on the following criteria: 1) choose three symmetrical figures [B, D, and E] from the given five figures for the first question, and 2) determine the correct valuation of at least three of the four propositions raised for the second question.

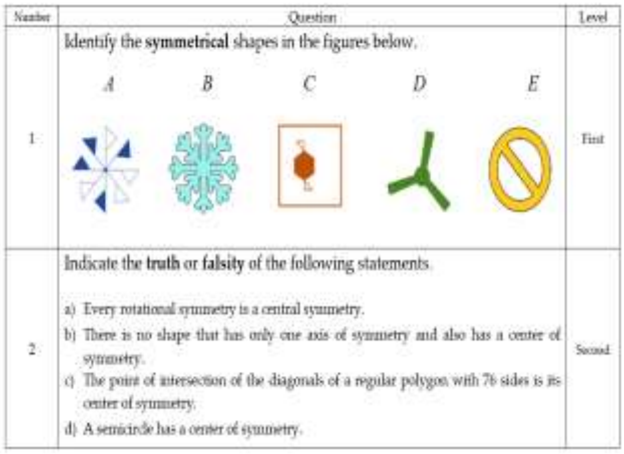


Fig. 4: Test questions based on van Hiele's theory.

The evaluation of the PTs group achievement at Van Hiele levels was qualitative based on the grading system proposed by [60]. Fig. 5 illustrates both qualitative and quantitative scales used to measure the process of achieving a level. This section can be divided into subheadings to provide a clear and organized description of the experimental results, their interpretation, and the conclusions drawn from the study. The section should be concise and precise, allowing readers to understand research outcomes clearly.

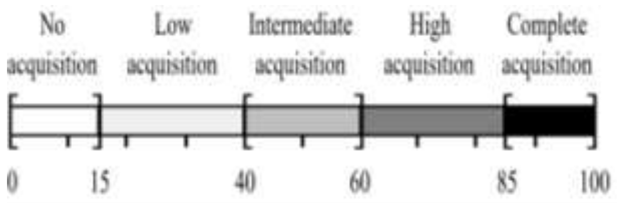


Fig. 5: Degrees of acquisition of a van Hiele level defined by [60, 61].

Test question of PMETs

To assess the cognitive perception of symmetric images, a test consisting of 5 symmetric images at Van Hiele level 2 (Analysis) was designed and administered to PMETs.

Stimuli used in the PMETs test

The images selected for this cognitive test were categorized into two groups: the first category is called Monosymmetric and includes images with a single type of symmetry, while the second category is called polysymmetric and includes images with two types of symmetry. In Fig. 6, Monosymmetric images are those with only one type of rotational (a), central (b), or axial (c) symmetry, while polysymmetric images include shapes with two types of axial-rotational (d) or axial-central (e) symmetry. Symmetries with 360-degree rotation (such as those naturally present in all 5 images) were disregarded.

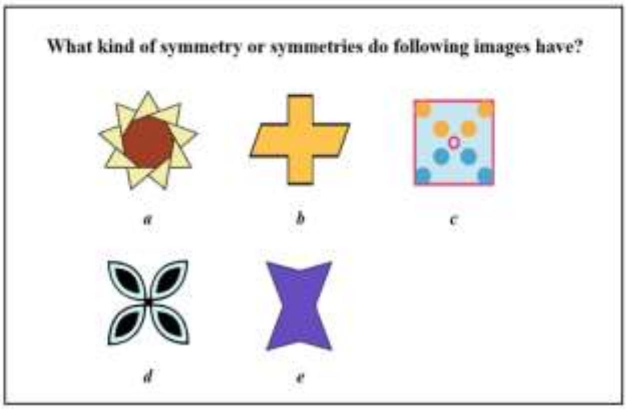


Fig. 6: The test question is based on the second level of Van Hiele's theory.

Procedure

The mixed quantitative-qualitative survey method was utilized to collect the data for this study. The measurement tool comprised two tests designed based on the Van Hiele theory and adapted from the levels of Van Hiele as presented by Soon [35]. The reliability of the test was assessed through Cronbach's alpha coefficient, which was found to be 0.68. The validity of the test was verified by mathematics professors.

The Eye Tracker device and cognitive science methods were employed to identify the type of symmetry in the designed images at the second level of the Van Hiele theory.

Procedure used in the PMETs test

Each trial began with a white fixation cross (+) on a gray background at the center of the screen (17'' CRT; refresh rate, 75 Hz; screen resolution 1920×1080; viewing distance 75 cm) for 500 milliseconds (ms). Then, questions and options were displayed for 500ms each. Finally, a symmetrical stimulus was shown. Participants viewed the stimulus and selected their answer from three options (rotational, central, and axial symmetry) with a mouse click on that option. Their eye movements were recorded throughout the task. One block of trials was collected from each participant, containing five trials of all symmetrical stimuli as shown in Fig. 6 (each presented once). Each block took 10 to 15 minutes to complete. See Figure 7 for a diagram of the paradigm. The task was implemented using the Psychophysics toolbox of MATLAB b2016 software [62, 63].

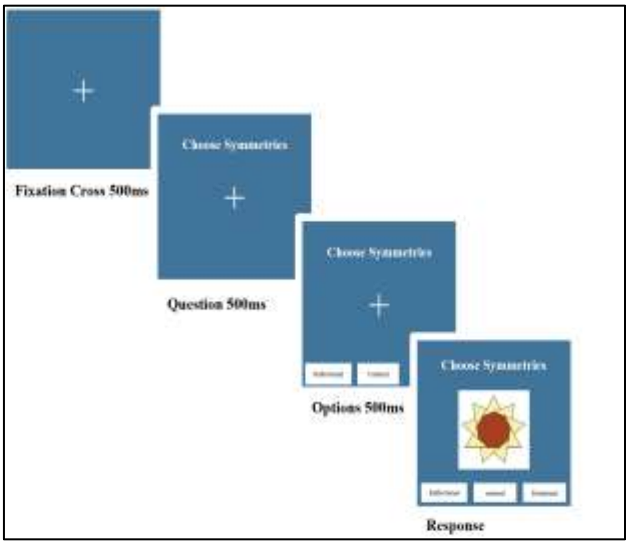


Fig. 7: Details of the test implementation in the computer laboratory of Shahid Rajaee Teacher Training University.

Eye Monitoring

Researchers used a non-invasive infrared eye-tracking device called Eyelink 1000 to monitor the participants' left-eye gaze positions during the study. The system measured the reflection of the pupil and cornea, with an accuracy of 0.25-0.75 visual degree. The researchers tracked the participants' eye movements to ensure that they were focused on the stimuli and to obtain their eye position.

Results and Findings

In this part, the research questions will be answered according to the analysis of the obtained data and the separation of the sample groups.

Results of PETs

The analysis of the data collected from the responses of PETs belonging to the PGT and NPGT groups to answer the first research question is presented in Table 3. In response to the second research question, the results of assignments conducted based on the questions and the first two levels of the van Hiele will be presented and analyzed.

Table 3: The percentage of correct answers to the questions and the level of qualitative achievement of van Hiele levels.

Sample	Question number	Frequency	Percent%	Qualitative acquisition of van Hiele levels
NPGT	1	11	26.19	Low acquisition
	2	6	14.28	No acquisition
PGT	1	6	75	High acquisition
	2	3	37.5	Low acquisition
Total	1	17	34	Low acquisition
	2	9	18	No acquisition

It can be concluded that PETs have the minimum necessary characteristics to achieve the first level of the Van Hiele theory, but they have not obtained the minimum characteristics necessary to achieve the second level of the Van Hiele theory. The details of this claim are shown in Table 3 and in summary, Fig. 8 illustrates the respondents' success rate percentage in answering the test questions.

The first question of the test and level one of van Hiele

The NPGT participants demonstrated a performance rate of 26.19%, falling in the low range of the first level of van Hiele, whereas the PGT participants scored 75%, falling in the medium range of the first level, see Table 4. All participants demonstrated success in identifying asymmetric Figs (A and C). In regards to the recognition of rotationally symmetric figures, all participants identified the snowflake (B) as rotationally symmetric but had a lower level of success in identifying the other two symmetrical figures. The causes of this difference resulted in the emergence of the idea of investigating geometric thinking based on cognitive science methods and designing and implementing the second test.

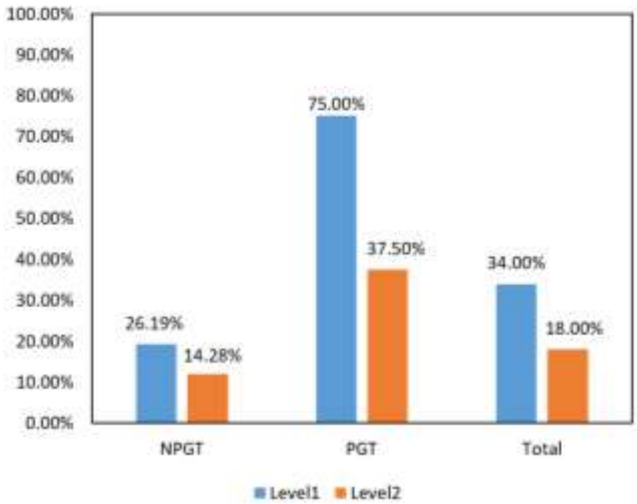


Fig. 8: The percentage of correct answers to van Hiele levels.

The second question of the test and the level of van Hiele

The Table 5 illustrates that the NPGT students were unable to reach the second level, with a performance rate of approximately 14%. Conversely, PGT students showed some improvement, scoring 37.5% in gaining a low level of the second van Hiele level theory. Interestingly, most of the errors made by the students were in response to questions related to the relationships between the line of symmetry and center of symmetry and relationships between rotational and central symmetry, affirming the significance of the research topic.

Results of PMETs

In the analysis of the data recorded by the Eye Tracker device (Eye Link100), the criteria listed in

Table 6 is taken into consideration for detecting the type of symmetry in the images displayed on the computer. The received response's scan path and heat maps were compared and analyzed with respect to the three groups considered.

Based on the successful performance of PETs in recognizing snowflake image as symmetrical figure, the researcher hypothesized that PMETs as an expert group would also excel in recognizing the rotational symmetry of Figs 6a or 6b. Therefore, the decision was made to conduct the cognitive science test, and its results demonstrated that this hypothesis was incorrect.

Fig. 9, which is an example of a scan path of completely correct and completely incorrect answers of students, according to the number of recorded saccades and the direction of eye movement in each image, shows that there is a significant difference between the two groups of answers so the scan path results confirm one of the research questions. This distinction is also evident in their response percentages (see Table 7).

Table 4: Percentage and frequency of correct identification of symmetrical and asymmetrical figures.

Sample		Figure number					based on the set criteria
		A	*B*	*C*	*D*	*E*	
NPGT	Frequency	42	42	42	10	19	11
	Percent%	100	100	100	23.80	45.23	26.9
PGT	Frequency	8	8	8	7	7	6
	Percent%	100	100	100	87.5	87.5	75
Total	Frequency	50	50	50	17	26	17
	Percent%	100	100	100	34	52	34

Table 5: Frequency and percentage of correct valuation of the second question.

Sample		Correct evaluation of the proposition				based on the set criteria
		A	*B*	*C*	*D*	
NPGT	Frequency	16	8	37	23	6
	Percent%	38.08	19.0	88.09	54.76	14.28
PGT	Frequency	3	7	3	3	3
	Percent%	37.50	87.50	37.50	37.50	37.50
Total	Frequency	19	15	40	26	9
	Percent%	38	30	80	52	18

Table 6: Criteria considered for response analysis.

Images	Completely Correct	Roughly correct	Ctelompley Incorrect
Monosymmetric	One correct choice without incorrect choice	The first choice is correct or incorrect and the second choice is incorrect or correct	Zero correct selection
Polysymmetric	Two correct choices without incorrect choices	Two or one correct choice	Zero correct selection

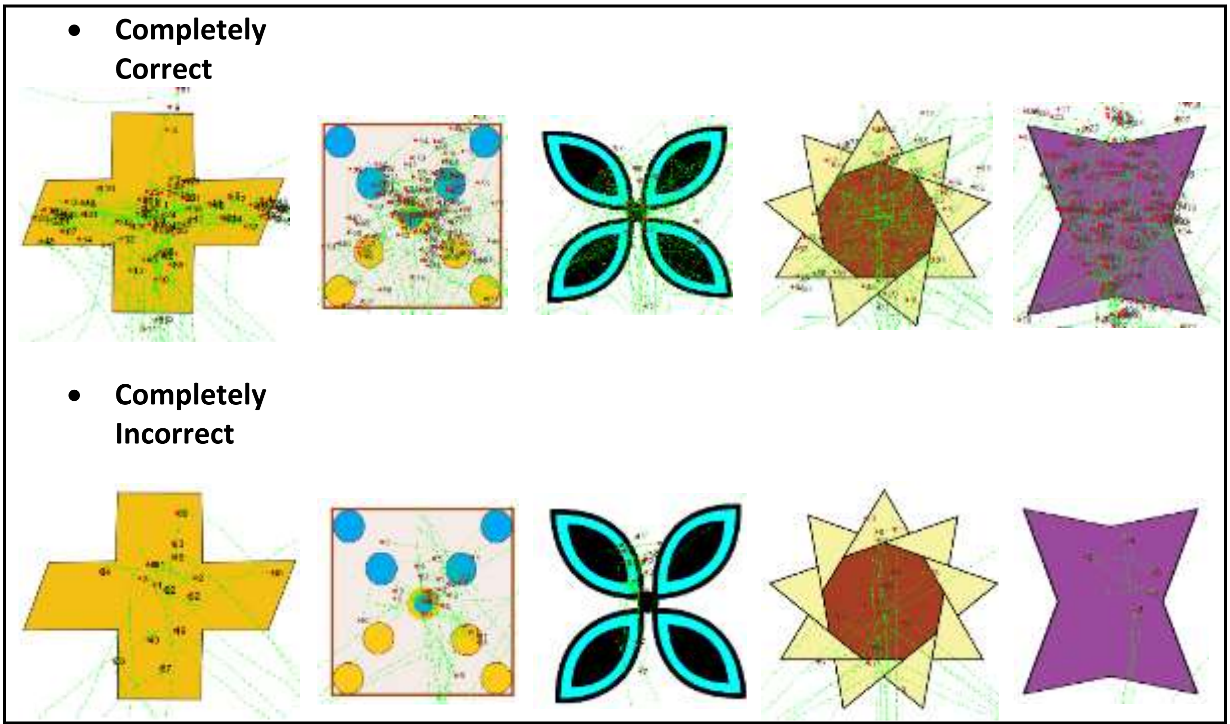


Fig. 9: Number of saccades and fixations in the scan path.

Table 7: Choosing the symmetry of the images according to the considered criteria.

Image	Symmetry type	subject	Correct %	Roughly correct %	Incorrect %
		Female	50	50	0
	Rotational	Male	26.66	46.66	26.66
		Total	33.33	47.61	19.04
		Female	50	3.33	16.66
	Central	Male	53.33	26.66	20
		Total	52.38	28.57	19.04
		Female	50	33.33	16.66
	Axial	Male	100	0	6.66
		Total	80.95	9.52	9.52
		Female	16.66	83.33	0
	Rotational Axial	Male	6.66	86.66	6.66
		Total	9.52	85.71	4.7
		Female	16.66	66.66	0
	Central Axial	Male	6.	20	6.66
		Total	47.61	47.11	4.7

Moreover, the scan path and heat map images reveal that certain participants' inability to provide accurate responses can be attributed to factors such as inattentiveness, inadequate attention to detail, and incomplete observation of the images. On the other hand, the outcomes derived from participants' heat map images validate the findings of previous research [64], indicating that reflections with vertical or horizontal mirror lines are generally more accessible for students to visualize compared to reflections with diagonal mirror lines.

The counts and percentages of selected symmetry types for each image in Table 8 reveal that, across all images, PMETs with an accuracy exceeding 60% were capable of correctly identifying image symmetries. However, their accuracy diminished in their final responses. In certain cases, their second or third choices were made incorrectly, resulting in inaccurate answers to the questions.

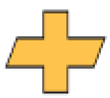
The difference in choosing the central symmetry option between image b (monosymmetric) and image e (polysymmetric) is also evident, with percentages of 80.95% and 76.19%, respectively. Notably, the disparity in choosing rotational symmetry between monosymmetric and polysymmetric images is 5 times that observed for central symmetry. These variations are further validated by the heat map images. For instance, the correct answer image in heat map Figure 10a displays a higher number of saccades compared to Fig. 11d, affirming these discrepancies.

On the other hand, in the two polysymmetric images (d) and (e), the linear symmetry option has been chosen by the participants with close percentages of 80% and 85%, respectively. In contrast, the detection percentage of linear symmetry in the monosymmetric image (c) increased to approximately 95%.

The sum of these results, together with the results of the order of correctly recognizing the symmetry of the images (axial, central, central axial, rotational, and finally rotational axial), show that the students' skill in distinguishing the type of symmetry in monosymmetric images is better than in poly symmetric images.

Fig. 10-11 show heat map images of PMETs tests according to the order of selection of the options in the test. It should be noted that the range of colors from cool to warm tones shown in the following images indicates the least to most focal points emphasized. Below is a summary of the findings derived from these images.

Table 8: Choosing symmetry in each image.

Image	Sample	Axial		Central		Rotational	
		f	%	*f*	%	*f*	%
	Female	0	0	3	50	6	100
	Male	3	20	9	60	12	80
	Total	3	14.28	12	57.14	18	85.71
	Female	0	0	5	83.33	3	50
	Male	5	33.33	12	80	3	20
	Total	5	23.80	17	80.95	6	28.57
	Female	5	83.33	1	16.66	2	33.33
	Male	15	100	0	0	1	6.66
	Total	20	95.23	1	4.76	3	14.28
	Female	5	83.33	5	83.33	5	83.333
	Male	13	86.66	10	66.66	8	53.33
	Total	18	85.71	15	71.42	13	61.90
	Female	6	100	3	50	4	66.66
	Male	12	80	13	86.66	4	26.66
	Total	17	80.95	16	76.19	8	38.09

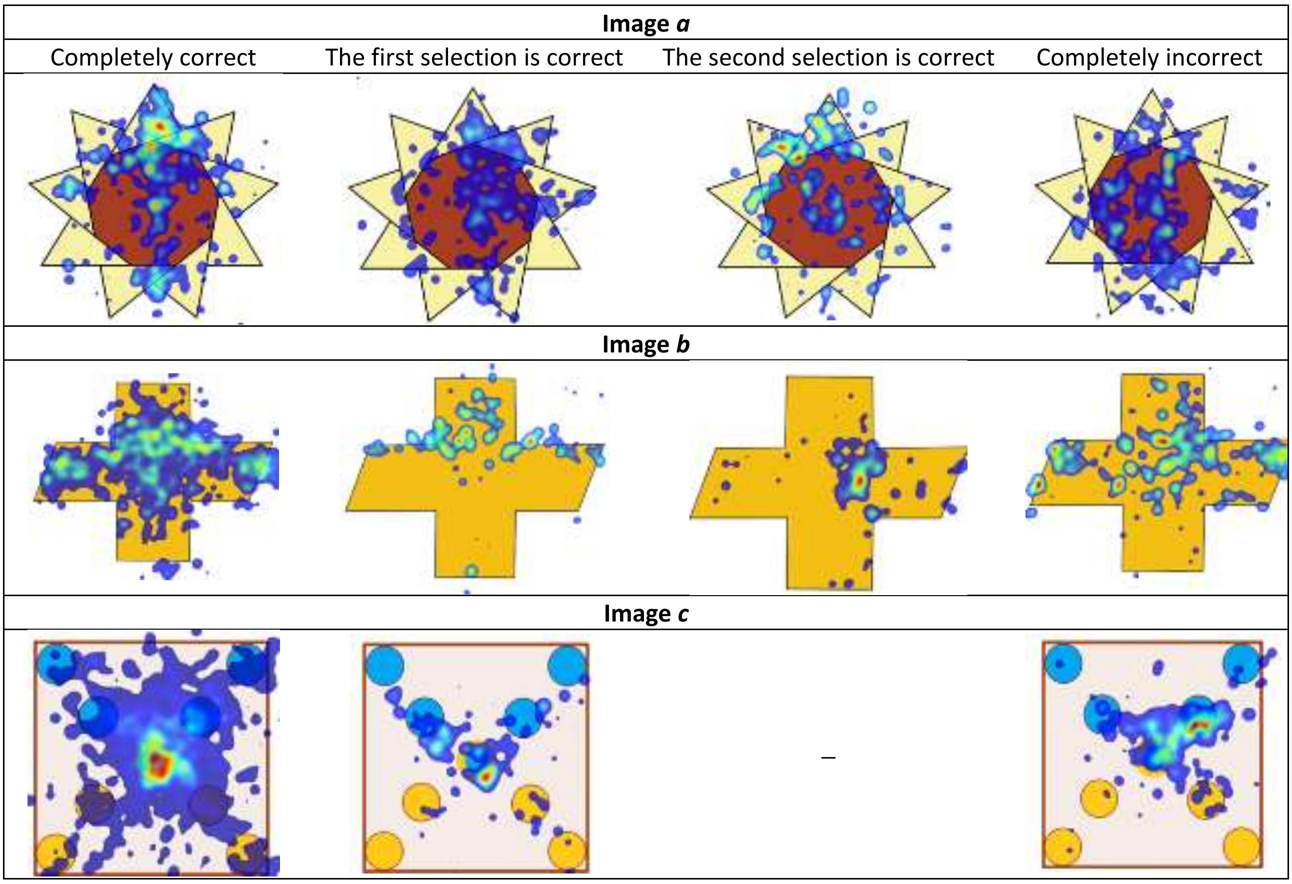


Fig. 10: Heat map monosymmetric images.

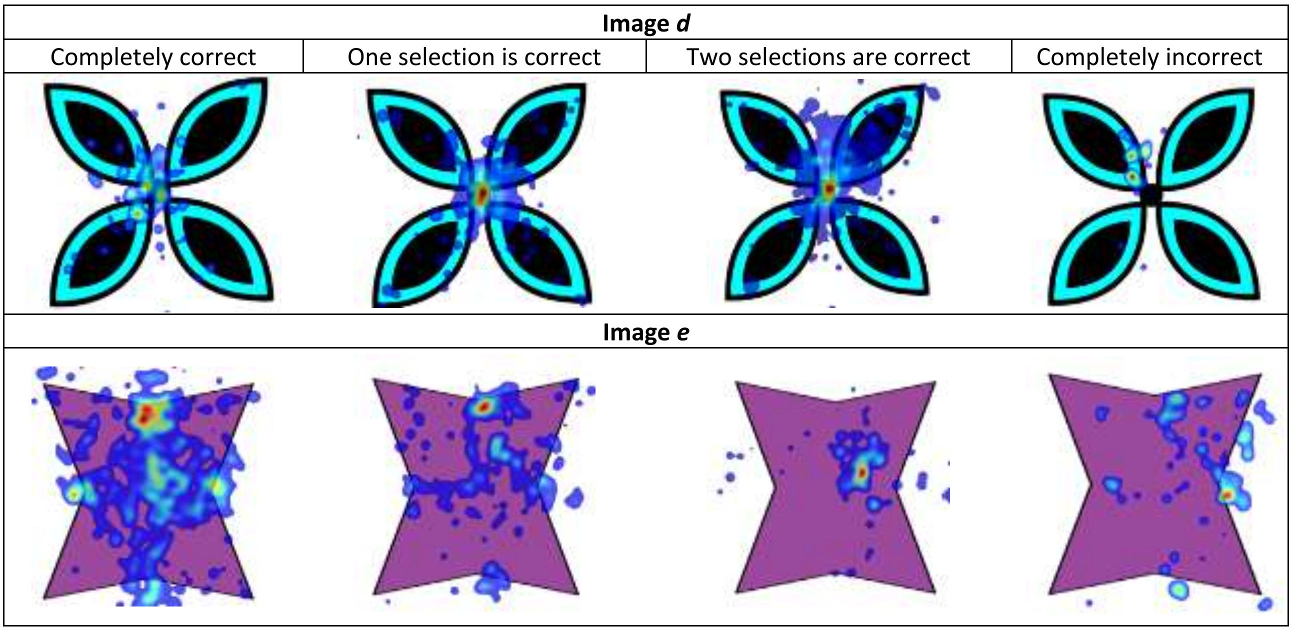


Fig. 11: Heat map polysymmetric images.

The results of these images follow a similar pattern to the scan path results. Images with 'completely correct' answers have more color points and warmer colors in the spectrum. In specific instances, the gaze trace and the warmer color spectrum indicate the determination of central or rotational symmetry, as well as the direction of rotation and the presence of axial symmetry (be it vertical, horizontal, or diagonal). The variable number of saccades can indicate the level of accuracy and concentration, the complexity of the image, or the recognition and understanding of the participants in determining the type of symmetry of the test images.

It can be seen that the number of saccades in each of the categories ‘completely correct’ to ‘completely incorrect’ in Fig. 10-11 ranges from the highest to the lowest. Of course, there is an exception in this process. In Fig. 10a, in contrast to the other cases, the number of saccades in the heat map of the category ‘completely incorrect’ is increased. One of the reasons is the weakness of the students in the group PEMT in understanding the concept of rotational symmetry correctly and completely, which causes more difficulty in recognizing and ultimately increasing the number of saccades and fixations. The results of Table 7 confirm this weakness. Among the 3 monosymmetric images, the lowest percentage of success (33.33%) is related to the image (a). Also, in polysymmetric images, the image (d) which has rotational-axial symmetry has a much lower success rate than the image (e) with central-axial symmetry.

Based on the distribution of colored dots in the images below, the research literature is corroborated by the fact that students in this experiment exhibited a greater emphasis on identifying vertical and then horizontal lines, whereas they rarely distinguished the oblique symmetry lines. According to the responses of the participants to the test, the lowest percentage of completely correct responses is related to Fig. 10d (9.52%), which displays the fewest colored dots among all recorded images.

Discussion

According to the analysis of the data obtained in response to the research questions, the following conclusions can be drawn:

The response rate for PET group students was 34%, which represents a low degree of achievement from the first level of Van Hiele's theory. According to Gutiérrez et al. [60, 61] scaling, they failed to achieve the minimum degree of achievement of the second level.

The discrepancy in Van Hiele levels noted between the NPGT and PGT groups confirms the effectiveness of the educational curriculum centered on geometric concepts during the academic of PETs. This elevation in their geometric thinking level highlights the beneficial outcomes of the educational program on their cognitive growth in this domain.

In both PET and PMET groups, students can recognize symmetrical shapes correctly but have difficulty determining the type of symmetry of shapes, especially when they encounter polysymmetric shapes or oblique or rotational axial symmetry. The results of the cognitive test on the PMET group support this conclusion. It has been found that most students of both two groups do not correctly understand rotational and central symmetry, as well as their characteristics.

There is a significant difference between each image and each type of symmetry of the symmetrical shapes based on heat map images. Students of PMETs who exhibited greater attentiveness, as evident from their scan path images with higher numbers of saccades and fixations, demonstrated better performance in accurately identifying symmetry.

The analysis of the data in Table 8 demonstrates that PMETs with an accuracy exceeding 60% were generally effective in identifying image symmetries, although their accuracy slightly declined in their final responses. Notably, the selection of central symmetry differed significantly between monosymmetric (image b) and polysymmetric (image e) images, with percentages of 80.95%

and 76.19%, respectively. The disparity in recognizing rotational symmetry between these image types was even more pronounced, highlighting students' superior skill in identifying symmetry types in monosymmetric images, particularly linear symmetry (approximately 95% accuracy in image c). These findings emphasize the importance of further investigating students' proficiency in perceiving different symmetry types.

Comparing the performance of PEMT students in the 'Roughly correct' and ' Completely correct ' criteria reveals that they are more prone to errors when precise determination of symmetry type is necessary. This pattern was consistently observed in the results obtained from the PET group, which also exhibited subpar performance at the analysis level within the framework of Van Hiele's theory.

Notable differences emerged between the performances of males and females within the PEMTs. Specifically, girls exhibited greater proficiency in identifying rotational symmetry, whereas boys excelled in detecting central symmetry. Additionally, a significant contrast was observed in correctly identifying linear symmetry, with boys achieving a 100% accuracy rate compared to girls' 50% accuracy. These performance discrepancies underscore the presence of distinct patterns in geometric thinking between males and females.

Figs 10,11 present heat map images of the PMETs test. The color spectrum, ranging from cool to warm tones, signifies the emphasis from least to most focal points. The results from these images closely resemble the patterns observed in scan paths, where 'completely correct' responses exhibit warmer tones and more color points. These patterns often unmistakably indicate the determination of central or rotational symmetry, as well as the direction of rotation and the presence of axial symmetry (be it vertical, horizontal, or diagonal). The varying number of saccades serves as an indicator of accuracy, concentration, image complexity, or participants' recognition and understanding of the test image's symmetry type.

Conclusions

Geometric transformations as a part of geometry, which is one of the most challenging topics in school mathematics, was chosen as the subject of this research. The current research seeks to find the root of students' problems in learning this subject, hence the statistical population was selected from prospective teachers who are one of the primary and effective factors in the education of students. It is obvious that the educational topics in the teacher training course are important and have an impact on their teaching skills, and the results of the research confirm that the existence of the topic of geometric transformations in the current curriculum of prospective teachers is necessary, but not sufficient and needs to be revised. Also, the authors used cognitive methods to observe the roots of students' answers in the designed test.

The heat map analysis of PMETs test results revealed distinct patterns in symmetry recognition across different types of symmetrical shapes. Students exhibiting heightened attentiveness, reflected in increased saccades and fixations in their scan paths, demonstrated improved accuracy in identifying symmetrical patterns. Noteworthy differences were observed in selecting central and rotational symmetries, particularly in monosymmetric images, highlighting the importance of investigating students' proficiency in perceiving various symmetry types.

Overall, both groups of prospective teachers demonstrated a lack of success in comprehending and identifying various types of symmetry, particularly the non-expert group. It is advisable to implement essential revisions in the development of educational content and teaching methodologies employed by professors, aiming to enhance students' grasp of this subject. Moreover, leveraging technology such as augmented reality (AR), GeoGebra, and Cabri Geometry, along with online platforms like mathsisfun.com and brilliant.org, can significantly contribute to the improvement of cognitive and spatial abilities among both students and teachers in this domain. By incorporating these resources, educators can foster a more effective and engaging learning experience, ultimately yielding better outcomes in the understanding of geometric transformations and symmetry concepts.

There are potential factors that may contribute to the different patterns of geometric thinking observed, such as individual aptitude, previous experiences, or educational background. Moreover, it is imperative to promote an inclusive and equitable learning environment where all students, regardless of gender, have equal opportunities to excel in mathematics and related subjects. Further research on these factors or finding a stable mental map of thinking about geometric transformations can be future research topics.

Authors' Contribution

All authors have the same contribution.

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Conflicts of Interest

The authors have no conflicts of interest.

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